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# Note on Galois extensions

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Throughout, $A$ will represent a ring with 1, and $B$ a subring of $A$ containing 1. Let $V$ be the centralizer $V_A(B)$ of $B$ in $A$, $H = V_A(V)$, and $C$ the center of $A$.

The present note contains several results concerning Galois extensions, where an (Artinian) simple ring extension $A/B$ is called a [finite] <u>Galois extension</u> if [${}_BA$ is finitely generated and] $V$ is a simple ring and $B$ coincides with the fixring of the group $G$ of all $B$-ring automorphisms in $A$ (cf. [7]). A Galois extension $A/B$ is said to be <u>outer</u> Galois or <u>inner</u> Galois according as $V = C$ or $H = B$. If $A/B$ is finite outer Galois, then it is known that $A/B$ is $\tilde{G}$-Galois (cf. [7, Proposition 9.6]). As to other terminologies and notations used in this note, we follow [2] and [7].

First, we shall prove the following which will enrich the substance of the theory of separable extensions:

<u>Theorem 1</u>. If $A/B$ is finite Galois then it is a separable extension.

Proof. Since $A/H$ is finite inner Galois, we have $\mathrm{End}\, A_H = A_L \cdot V_R$. Noting that $[\mathrm{Hom}({}_HA, {}_HH):H]_R = [A:H]$, we readily see that

$$ {}_AA \otimes_H A_A \cong {}_A\mathrm{Hom}(H_H, A_H) \otimes_H A_A \cong {}_A\mathrm{Hom}(\mathrm{Hom}({}_HA, {}_HH)_H, A_H)_A $$

$$ \cong {}_A(\mathrm{End}\, A_H)_A \cong {}_A(A \oplus \cdots \oplus A)_A \ . $$

Hence, ${}_AA \otimes_H A_A$ is homogeneously completely reducible and the $A$-$A$-homomorphism $A \otimes_H A \longrightarrow A$ $(a \otimes a' \longmapsto aa')$ splits. On the other hand, $H/B$ being finite outer Galois, by [1, Proposition 3.3] the

H-H-homomorphism $H \otimes_B H \longrightarrow H$ $(h \otimes h' \longmapsto hh')$ splits, whence it follows that the A-A-homomorphism $A \otimes_B A \longrightarrow A \otimes_H A$ $(a \otimes a' \longmapsto a \otimes a')$ splits. Combining those above, we readily see that the A-A-homomorphism $A \otimes_B A \longrightarrow A$ $(a \otimes a' \longmapsto aa')$ splits.

If $A/B$ is H-separable, i.e., if ${}_A A \otimes_B A_A < \oplus\ {}_A(A \oplus \dots \oplus A)_A$, then there exist some $v_i \in V$ and $\sum_j x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A = \{u \in A \otimes_B A \mid au = ua \text{ for all } a \in A\}$ such that $\sum_{i,j} x_{ij} \otimes y_{ij} v_i = 1 \otimes 1$. For $g \in \mathrm{End}\, A_B$, $\sum_{i,j} x_{ij} \otimes y_{ij} a v_i = a \otimes 1$ implies then $\sum_{i,j} g(x_{ij}) \otimes y_{ij} a v_i = g(a) \otimes 1$, whence it follows $\sum_{i,j} g(x_{ij}) y_{ij} a v_i = g(a)$ (cf. [3]). Especially, we have

Proposition 1. If $A/B$ is H-separable, then $\mathrm{End}\, A_B = A_L \cdot V_R$.

As was shown in [8, Proposition 1.1], if $A$ is a separable R-algebra and a projective R-module then $A$ is a finitely generated R-module. We state here an analogue of the above for H-separable extensions.

Proposition 2 (cf. [6]). If $A/B$ is H-separable and $A_B$ is projective then $A_B$ is finitely generated.

Proof. Let $\{a_\kappa ; f_\kappa\}_{\kappa \in K}$ $(a_\kappa \in A,\ f_\kappa \in \mathrm{Hom}(A_B, B_B))$ be a projective coordinate system for $A_B$. Then, $f_\kappa$ extends naturally to $f_\kappa^* \in \mathrm{Hom}(A \otimes_B A_A, A_A)$ and $\{a_\kappa \otimes 1, f_\kappa^*\}_{\kappa \in K}$ is a projective coordinate system for $A \otimes_B A_A$. Since $A \otimes_B A_A$ is finitely generated, we can find a finite subset $K'$ of $K$ such that $A \otimes_B A = \sum_{\kappa \in K'} (a_\kappa \otimes 1)A$. We consider here the set $K'' = \{\kappa \in K \mid f_\kappa(a_{\kappa'}) \neq 0 \text{ for some } \kappa' \in K'\}$, which is obviously a finite subset of $K$.

If $a$ is an arbitrary element of $A$ then $\{\kappa \in K \mid f_\kappa^*(a\otimes 1) \neq 0\} \subseteq K''$ and we have

$$a\otimes 1 = \sum_{\kappa\in K}(a_\kappa\otimes 1)f_\kappa^*(a\otimes 1) = \sum_{\kappa\in K''}(a_\kappa\otimes 1)f_\kappa^*(a\otimes 1)$$

$$= \sum_{\kappa\in K''} a_\kappa f_\kappa(a)\otimes 1 ,$$

which implies $a \in \sum_{\kappa\in K''} a_\kappa A$.

The next is only a combination of [4, Theorem 1.5] and [5, Theorem 2.1]. However, Propositions 1 and 2 and the proof of Theorem 1 enable us to obtain a shorter proof.

<u>Theorem 2</u>. If $B$ is simple, then the following conditions are equivalent:

(1) $A/B$ is finite inner Galois.

(2) ${}_A A\otimes_B A \cong {}_A(A\oplus \cdots \oplus A)_A$ .

(3) $A/B$ is H-separable.

Proof. As (1) $\Longrightarrow$ (2) is obvious by the proof of Theorem 1 and (2) $\Longrightarrow$ (3) is trivial, it remains only to prove (3) $\Longrightarrow$ (1). By Proposition 2, $A_B$ is finitely generated projective. Hence, there exist $a_1, \ldots, a_n \in A$ and $f_1, \ldots, f_n \in \mathrm{Hom}(A_B, B_B) \subseteq \mathrm{End}\, A_B = A_L\cdot V_R$ (Proposition 1) such that $a = \sum_i a_i f_i(a)$ for any $a \in A$. If $I$ is an arbitrary non-zero ideal of $A$ then $f_i(I) \subseteq AIV \cap B = I\cap B$, and so $I \subseteq \sum_i Af_i(I) \subseteq A(I\cap B)$, namely, $I = A(I\cap B) = AB = A$. This means that $A$ is simple. Moreover, the simplicity of $\mathrm{End}\, A_B = A_L\cdot V_R$ ($\simeq A\otimes_C V^o$) yields the simplicity of $V$ and $[V:C] = [A:B]_R$. Hence, $A/B$ is finite inner Galois.

<u>Remark 1</u>. Let $A$ be the $2\times 2$-matrix ring over a division ring $D$; $A = \sum_{i,j=1}^{2} De_{ij}$, and $B = De_{11}\oplus De_{22}$. Then, $A/B$ is a free

H-separable extension (but $A$ is not centrally projective over $B$, i.e., ${}_B A_B$ can not be a direct summand of ${}_B(B \oplus \dots \oplus B)_B$). In fact, $\{1, t = e_{12} + e_{21}\}$ is a right [left] free $B$-basis of $A$ and $\{1 \otimes e_{11} + t \otimes e_{12}, 1 \otimes e_{22} + t \otimes e_{21}\}$ is a free $A$-basis of $A \otimes_B A$ contained in $(A \otimes_B A)^A$. This will show that for a projective H-separable extension $A/B$ the simplicity of $A$ need not imply that of $B$.

Finally, we shall prove a slight improvement of [2, Theorem 3].

Theorem 3. Let $A$ be Galois and left algebraic over $B$. If $[V:C] < \infty$ then the following conditions are equivalent:

(1) ${}_B A_B$ is completely reducible.

(2) ${}_H A_H$ and ${}_B H_B$ are completely reducible.

(3) ${}_H H_H <\oplus\, {}_H A_H$ and ${}_B B_B <\oplus\, {}_B H_B$ .

(4) $V/C$ is separable and ${}_B B_B <\oplus\, {}_B H_B$ .

Proof. The equivalence of (1), (2) and (4) has been proved in [2, Theorem 3]. It suffices therefore to show that if ${}_H H_H <\oplus\, {}_H A_H$ then $V/C$ is separable. To be easily seen, End ${}_H A_H = V_L \cdot V_R$ is canonically V-V-isomorphic to $V \otimes_C V$. Hence, there exists an element $\sum_i v_i \otimes v_i' \in V \otimes_C V$ such that $\sum_i v_i v_i' = 1$ and $\sum_i v_i a v_i' \in H$ for all $a \in A$. If $v$ is in $V$ then $\sum_i v v_i a v_i' = \sum_i v_i a v_i' v$, which means $\sum_i v_i \otimes v_i' \in (V \otimes_C V)^V$. Accordingly, $V/C$ is separable.

As a special case of Theorem 3, we have the following which contains [5, Corollary 2 (2)]:

Corollary 1. Let $A$ be finite inner Galois over $B$. If $B'$ is a simple intermediate ring of $A/B$, then the following conditions are equivalent:

(1) ${}_{B'}A_{B'}$ is completely reducible.

(2) ${}_{B'}B'_{B'} <\oplus {}_{B'}A_{B'}$ .

(3) $V_A(B')/C$ is separable.

Remark 2. The following example will show that, under the hypothesis of Corollary 1, ${}_BB_B <\oplus {}_BA_B$ need not imply ${}_{B'}B'_{B'} <\oplus {}_{B'}A_{B'}$ : Let $P/\Phi$ be a two dimensional purely inseparable field extension. If we set $A = (\Phi)_2$ and $B = \Phi$ , then $A/B$ is finite inner Galois and there exists an intermediate ring $B'$ of $A/B$ which is $B$-ring isomorphic to $P$. By Corollary 1, ${}_{B'}A_{B'}$ is not completely reducible, but ${}_BA_B$ is obviously completely reducible.

Remark 3. Theorem 3 will form a contrast with [2, Theorem 4]: Let $A$ be Galois and left algebraic over $B$. If $[V:C] < \infty$ then the following conditions are equivalent:

(1) ${}_BA_B$ is completely indecomposable.

(2) ${}_HA_H$ and ${}_BH_B$ are completely indecomposable.

(3) i) $V = V_B(B)$ and is purely inseparable over $C$;

ii) either $A/B$ is inner Galois or the Galois group of $H/B$ is a pro-p-group and $A$ is of characteristic p.

## References

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